

2-subcoloring is NP-complete for planar comparability graphs

Pascal Ochem
CNRS - LIRMM, Montpellier, France

February 7, 2017

Abstract

A k -subcoloring of a graph is a partition of the vertex set into at most k cluster graphs, that is, graphs with no induced P_3 . 2-subcoloring is known to be NP-complete for comparability graphs and three subclasses of planar graphs, namely triangle-free planar graphs with maximum degree 4, planar perfect graphs with maximum degree 4, and planar graphs with girth 5. We show that 2-subcoloring is also NP-complete for planar comparability graphs with maximum degree 4.

1 Introduction

A k -subcoloring of a graph is a partition of the vertex set into at most k cluster graphs, that is, graphs with no induced P_3 . Unlike k -coloring, k -subcoloring is already NP-complete for $k = 2$:

Theorem 1. *2-subcoloring is NP-complete for the following classes:*

- (1) $(K_4, \text{bull, house, butterfly, gem, odd-hole})$ -free graphs with maximum degree 5 [1],
- (2) triangle-free planar graphs with maximum degree 4 [2, 3],
- (3) $(K_{1,3}, K_4, K_4^-, C_4, \text{odd-hole})$ planar graphs [4],
- (4) planar graphs with girth 5 [7].

A graph G is (d_1, \dots, d_k) -colorable if the vertex set of G can be partitioned into subsets V_1, \dots, V_k such that the graph induced by the vertices of V_i has maximum degree at most d_i for every $1 \leq i \leq k$. Notice that every $(1, 1)$ -colorable graph is 2-subcolorable. Moreover, on triangle-free graphs, $(1, 1)$ -colorable is equivalent to 2-subcolorable. As it is well known, for every $a, b \geq 0$, every graph with maximum degree $a + b + 1$ is (a, b) -colorable [6]. Thus, every graph with maximum degree 3 is 2-subcolorable, so that the degree bound of 4 in Theorems 1.(2) and 1.(3) is best possible.

Notice that the graphs in Theorem 1.(1) are comparability graphs since they are (bull, house, odd-hole)-free [5]. Our main result restricts the class in Theorem 1.(1) to planar graphs and lowers the maximum degree from 5 to 4.

Theorem 2. *Let \mathcal{G} denote the class of $(K_4, \text{bull, house, butterfly, gem, odd-hole})$ -free planar graphs with maximum degree 4. 2-subcoloring is NP-complete for \mathcal{G} .*

2 Main result

The reduction is from 2-subcoloring (or equivalently $(1, 1)$ -coloring) on triangle-free planar graphs with maximum degree 4, which is NP-complete by Theorem 1.(2). From an instance graph G of this problem, we construct a graph G' in \mathcal{G} . Every vertex v of G is replaced by a copy H_v of the vertex gadget H depicted in Figure 1. For every edge uv of G , we use two copies of the edge gadget E depicted in Figure 2 to connect H_u and H_v as follows:

- We identify the vertex x_1 of the first (resp. second) edge gadget with a vertex a_{2p} (resp. a_{2p+1}) of H_u , with $0 \leq p \leq 3$.
- We identify the vertex x_2 of the first (resp. second) edge gadget with a vertex a_i (resp. a_j) of H_v such that $\min(i, j) \equiv 0 \pmod{2}$, $|j - i| = 1$, and no edge crossing is created.

It is easy to check that G' can be made planar and with maximum degree 4. Moreover, G' is $(K_4, \text{bull, house, gem, butterfly})$ -free. By removing the vertices whose neighborhood induces a P_3 , we obtain a bipartite graph. This shows that G' is odd-hole free. Thus G' belongs to \mathcal{G} .

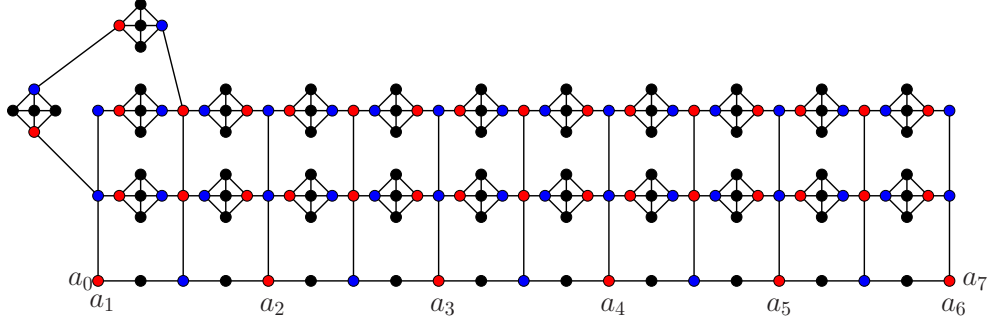


Figure 1: The vertex gadget H .

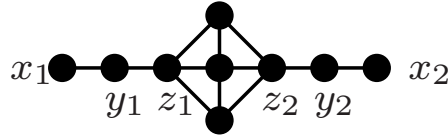


Figure 2: The edge gadget E .

Let us show that G' is 2-subcolorable if and only if G is 2-subcolorable. Given a 2-subcoloring of a graph, we say that a vertex p is *saturated* if there exists a monochromatic edge pq and is unsaturated otherwise. We will need the following properties of E :

1. In every 2-subcoloring of $E \setminus \{x_1, x_2, y_1, y_2\}$, the vertices z_1 and z_2 get distinct colors and are saturated.
2. In every 2-subcoloring of $E \setminus \{x_1, x_2\}$, the vertices y_1 and y_2 get distinct colors and are not saturated.
3. There exists a 2-subcoloring of E such that the vertices x_1 and x_2 get distinct colors and are not saturated.
4. In every 2-subcoloring of E such that the vertices x_1 and x_2 get the same color, exactly one vertex in $\{x_1, x_2\}$ is saturated.

The six vertices labeled a_i in H are called *ports*. Using properties (1) and (2), we obtain that every 2-subcoloring of H (in colors red and blue) forces the color of many vertices (see Figure 1). In particular, all the ports in H get the same color. This common color is said to be the color of H_v corresponds to the color of v in a 2-subcoloring of G . We also check that in every 2-subcoloring of H , at most one of the ports is not saturated.

Suppose that uv is an edge in G . Consider the 2-subcolorings of the subgraph of G' induced by H_u , H_v , and the two edge gadgets for the edge uv . If distinct colors are given to H_u and H_v , then

this 2-subcoloring can be extended to the edge gadgets using property (3). Since this extension does not saturate any of the considered ports of H_u and H_v , H_u can be connected to any number of vertex gadgets with the color distinct from the color of H_u . If the same color is given to H_u and H_v , then this 2-subcoloring can be extended using property (4). However, this coloring extension saturates the unique unsaturated port in both H_u and H_v . Thus, H_u can be connected to at most one vertex gadget with the same color as H_u .

This shows that G' is 2-subcolorable if and only if G is 2-subcolorable.

References

- [1] H. Broersma, F.V. Fomin, J. Nešetřil, G. Woeginger. More About Subcolorings, *Computing* **69** (2002), 187–203,
- [2] J. Fiala, K. Jansen, V.B. Le, and E. Seidel. Graph Subcolorings: Complexity and Algorithms, *SIAM J. Discrete Math.* **16(4)** (2003), 635–650.
- [3] J. Gimbel and C. Hartman. Subcolorings and the subchromatic number of a graph, *Discrete Math.* **272** (2003), 139–154.
- [4] D. Gonçalves and P. Ochem. On star and caterpillar arboricity. *Discrete Math.* **309(11)** (2009), 3694–3702.
- [5] <http://www.graphclasses.org>. Information System on Graph Classes and their Inclusions (ISGCI) by H.N. de Ridder et al. 2001-2014
- [6] L. Lovász. On decomposition of graphs. *Studia Sci. Math. Hungar* **1** (1966), 237–238.
- [7] M. Montassier and P. Ochem. Near-colorings: non-colorable graphs and NP-completeness. *Electron. J. Comb.* **22(1)** (2015), #P1.57.